LOGIC

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GENERALIZING PRIKRY FORCING AND PARTITION RELATIONS

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In [6] Prikry introduced a new forcing notion. Using a normal measure $\mathcal{U}$ on κ, forcing by P adds and ω-sequence τ, cofinal in κ. The forcing notion P has the following properties:

(i) The direct extension lemma: For every sentence φ, every condition $p \in P$ can be extended to $q \in P$, without imposing new elements on τ, and q decides φ.

(ii) An ω-sequence τ, cofinal in κ, is generic over P iff every $X \in \mathcal{U}$ contains a final segment of τ.

(iii) If τ is generic over P, then every subsequence of τ is generic over P.

Using higher measurable cardinals for which $o(\kappa) < \kappa$ Magidor [3] has generalized Prikry's forcing to introduce longer closed sequences cofinal in κ. Let Mitchell [4] has shown that Magidor's procedure can be generalized whenever $o(\kappa) \leq \kappa^+$. But Magidor forcing shares only property (i) from Prikry forcing. Then Radin [7] has generalized Magidor forcing further, using highly closed-elementary embeddings. His forcing has property (i) too and property (iii) if κ is hyper measurable. Using κ much weaker than $o(\kappa) = \kappa$, we introduce a forcing notion with a "short" generic sequence closed unbounded in κ (short means: of order type ω^n $(n < \omega)$) and the forcing notion P has all properties (i)-(iii) as Prikry forcing.

Then we apply our forcing notion to prove some consistency results concerning partition relations. To state our theorems we need some definitions:

Definition 0.1. $o^n(\kappa)$ (the n^{th} measurability order of κ) is defined by induction on $n < \omega$:

$o^0(\kappa) = \kappa$.

$o^{n+1}(\kappa) = o\big(o^n(\kappa)\big)$ ($o(\mu)$ is the usual Mitchell's order of measurability).

Definition 0.2. $\kappa \underset{\text{def}}{\longrightarrow} (\mu)^{\alpha}_{\lambda}$ ($\kappa \underset{\text{def}}{\xrightarrow{\text{Cl}}} (\mu)^{\alpha}_{\lambda}$) stands for

"Every definable function $f : [\kappa]^{\alpha} \longrightarrow \lambda$ has a homogeneous sequence (which is closed) of order type μ".

Our main results are

Theorem 1. *Let n be a natural number, $n \geq 2$.*
If $\mathrm{Con}(ZFC + \exists\kappa(o^n(\kappa) = 2))$ *then* $\mathrm{Con}(ZFC + \aleph_1 \underset{\text{def}}{\xrightarrow{\text{Cl}}} (\omega^n)^{\omega^n}_{\aleph_0})$.

Theorem 2. *Let n be a natural number, $n \geq 1$.*
If $\mathrm{Con}(ZFC + \exists\kappa(o^n(\kappa) = 2))$ *then* $\mathrm{Con}(ZFC + \aleph_1 \underset{\text{def}}{\longrightarrow} (\omega^{n+1})^{\omega^{n+1}}_{\aleph_0})$.

Remarks.

a) The case for $n = 1$ in Theorem 2 has been proved in [2].

b) Spector [8] has proved in $\mathrm{Con}(\exists\kappa(o(\kappa) = 1))$ iff $\mathrm{Con}(\aleph_1 \underset{\text{def}}{\longrightarrow} (\omega)^{\omega}_{\aleph_0})$ and from [2] we known that $\mathrm{Con}(\exists\kappa(o(\kappa) = 2))$ iff $\mathrm{Con}(\aleph_1 \underset{\text{def}}{\longrightarrow} (\omega^2)^{\omega^2}_{\aleph_2})$. Our Theorem 2 is one direction toward the proof that there are jumps in the consistency strength of partition relations which deal with ω^n-sequence for $n > 2$.

c) In [1] we have proved that $\mathrm{Con}(\aleph_1 \underset{\text{def}}{\longrightarrow} (\omega^{\omega})^{\omega^{\omega}}_{\aleph_0})$ implies $\mathrm{Con}(\exists\kappa(o(\kappa) > 1))$, thus ω^{ω} is another jump in the pattern of the consistency strength.

References

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